

# PALINDROMIC BRAIDS

FLORIAN DELOUP, DAVID GARBER, SHMUEL KAPLAN, AND MINA TEICHER

ABSTRACT. The braid group  $B_n$ , endowed with Artin's presentation, admits an antiautomorphism  $B_n \rightarrow B_n$ , such that  $v \mapsto \bar{v}$  is defined by reading braids in reverse order (from right to left instead of left to right). We prove that the map  $B_n \rightarrow B_n$ ,  $v \mapsto v\bar{v}$  is injective. We also give some consequences arising due to this injectivity.

## 1. INTRODUCTION

Let  $n \geq 2$ . Any free group  $F_{n-1}$  on  $n-1$  generators  $\sigma_1, \dots, \sigma_{n-1}$  supports the antiautomorphism  $rev : w \mapsto \bar{w}$  defined by

$$\sigma_{i_1}^{\alpha_1} \cdots \sigma_{i_r}^{\alpha_r} \mapsto \sigma_{i_r}^{\alpha_r} \cdots \sigma_{i_1}^{\alpha_1},$$

which reverses the order of the word  $w$  with respect to the prescribed set of generators. It follows that any group  $G$  presented by generators and relations admits such an antiautomorphism  $rev$ . The elements of  $G$  which are order-reversing invariant are called *palindromic*. In this paper, we consider palindromic elements of Artin's Braid group  $B_n$ , equipped with Artin's presentation, which will be called *palindromic braids*. Artin's presentation of the braid group  $B_n$  consists of  $n-1$  generators  $\sigma_1, \dots, \sigma_{n-1}$  and relations

$$(1.1) \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2,$$

$$(1.2) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \leq i \leq n-2.$$

We distinguish between two equivalence relations on the elements of the braid group. For  $a, b \in B_n$  we write  $a = b$  to denote that  $a$  and  $b$  represent the same element in the group, and  $a \equiv b$  to denote that  $a$  and  $b$  are actually the same element written letter by letter (i.e.,  $a \equiv b$  means that  $a$  and  $b$  are equal in the free group using only the generators of the braid group, with no relators).

Palindromic braids have a particularly nice geometric interpretation. Given a geometric braid  $\beta$ , denote by  $\hat{\beta}$  its closure into a link inside a fixed solid torus  $D^2 \times S^1$ . The solid torus admits the involution

$$\text{inv} : D^2 \times S^1 \rightarrow D^2 \times S^1, \quad (re^{it}, \theta) \mapsto (re^{-it}, -\theta),$$

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whose set of fixed points consists of two segments ( $t \equiv 0 \pmod{\pi}$  and  $\theta \equiv 0 \pmod{\pi}$ ), which is the intersection of the axis of the  $180^\circ$  rotation with the solid torus. Observe that  $\widehat{rev(\beta)}$  is nothing else than  $\widehat{inv(\beta)}$  with the opposite orientation. In particular, if a braid  $\beta \in B_n$  is palindromic then  $\widehat{\beta}$  coincides with  $\widehat{inv(\beta)}$  with the opposite orientation, see Figure 1.1.

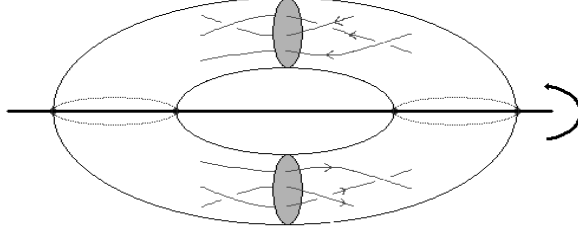


FIGURE 1.1. The involution  $inv$  and palindromic braids.

We prove the following rigidity result for palindromic braids.

**Theorem 1.1.** *Let  $\beta \in B_n$  be a palindromic braid such that*

$$(1.3) \quad \beta = v\bar{v}$$

*for some braid  $v \in B_n$ . Then the decomposition (1.3) is unique. Equivalently, let  $\beta = v\bar{v}$  and  $\beta' = v'\bar{v}'$  be two words in Artin's generators  $\sigma_1, \dots, \sigma_{n-1}$ . Then,  $\beta = \beta'$  in  $B_n$  if and only if  $v = v'$  in  $B_n$ .*

Of course one implication is obvious. Only the “only if” part of the statement deserves a proof.

*Remark 1.2.* Note that Theorem 1.1 cannot be generalized into the case of palindromic braid words of odd length. For example the two equal words  $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$  are of the form  $w\tau\bar{w}$  and  $v\sigma\bar{v}$ , however  $\sigma_1 = w \neq v = \sigma_2$ . Moreover, not all palindromic braids of even length are of the form (1.3). For example,  $\sigma_1\sigma_3 = \sigma_3\sigma_1$  however,  $\sigma_1 \neq \bar{\sigma}_3$ .

After this work was finished, F. Deloup communicated to us an alternative proof for Theorem 1.1, which is presented in [2], and is derived from the properties of the Dehornoy ordering of braids. The construction of the latter is a long process that requires rather sophisticated methods. In this paper we give an elementary proof, based on Garside normal form and its variant as developed by Jacquemard.

## 2. PRELIMINARIES AND THE JACQUEMARD ALGORITHM

This section is devoted to the building blocks we use in order to prove Theorem 1.1. Mainly, this section is intended to fix notations and recall some of the algorithms we use in this paper.

The monoid  $B_n^+$  of *positive braids* consists of braids which admit a word representative which does not contain  $\sigma_i^{-1}$ ,  $1 \leq i \leq n-1$ .

Among positive braids, we can consider those whose number of crossings between any two strands is less or equal to 1: they form the subset  $S_n^+ \subset B_n^+$  of *positive permutation braids*.

There is a canonical epimorphism  $B_n \rightarrow S_n$ . The image of a braid  $\gamma$  is the permutation *associated to*  $\gamma$ . In particular, it is known that  $S_n^+$  is in canonical bijection with the symmetric group  $S_n$ , which justifies the name of positive permutation braids.

There is only one positive braid  $\Delta \in B_n^+$  in which any pair of strings crosses exactly once. It corresponds geometrically to a generalized half-twist which consists of all the strands  $1, \dots, n$ , and is called the *Garside element*.  $\Delta$  is given by the formula:

$$(2.1) \quad \Delta = (\sigma_1 \sigma_2 \dots \sigma_{n-1}) (\sigma_1 \sigma_2 \dots \sigma_{n-2}) \dots (\sigma_1 \sigma_2) \sigma_1.$$

A basic result asserts that the center of  $B_n$  is generated by  $\Delta^2$ . Abelianization of  $B_n$  yields a canonical homomorphism  $B_n \rightarrow \mathbb{Z}$  which, when restricted to  $B_n^+$ , coincides with word length with respect to Artin generators. We denote by  $|\beta|$  the length of  $\beta \in B_n^+$ ; we have  $|\sigma_i| = 1$  and the trivial braid  $e$  is the only positive braid such that  $|e| = 0$ .

We recall the algorithm given by Jacquemard [5], which manipulates a positive braid word  $w \in B_n^+$  in order to write it using a given leading letter  $\sigma_i$ . The output of the algorithm is an equivalent positive braid word  $\sigma_i w' = w$  or an indication that no  $w'$  exists such that equality holds.

The basic nature of the algorithm is greedy. It starts by asking whether  $w \equiv \sigma_i w'$  and stops if it does. If not, it looks for  $\sigma_i$  inside  $w$ . In case  $\sigma_i$  is not one of the letters of  $w$ , the algorithm returns *false* which indicates non existence of  $w' \in B_n^+$  such that  $\sigma_i w' = w$ .

When the algorithm found the leftmost  $\sigma_i$  it works in two steps:

- (1) Switch  $\sigma_i$  with its left neighbor  $\sigma_j$  as long as  $|i - j| \geq 2$ . If  $\sigma_i$  becomes the first letter of the word we are done. However, in case that  $|i - j| = 1$ , the word is of the form  $w = w_0 \sigma_j \sigma_i w_1$ , and so in order to move  $\sigma_i$  to the left one must use the triple relation (1.2) between  $\sigma_j \sigma_i$  and the left most letter of  $w_1$ . If this is the case, the algorithm goes to step (2).
- (2) The algorithm calls itself recursively with the word  $w_1$  and the letter  $\sigma_j$ . Upon success of the recursive call the word looks like  $w = w_0 \sigma_j \sigma_i \sigma_j w'_1$  and therefore, we activate relation (1.2) on  $\sigma_j \sigma_i \sigma_j$  resulting with  $w = w_0 \sigma_i \sigma_j \sigma_i w'_1$ , and return to step (1). However, if the recursive call fails to extract  $\sigma_j$  to the left of  $w_1$  the algorithm returns *false*.

To finish this section, we recall another result on the decomposition of braids, due to F. A. Garside [4] and later refined by W. P. Thurston [7], and by E. A. Elrifai and H. R. Morton [3].

**Definition 2.1.** We say that a product  $\alpha_1 \dots \alpha_r$  satisfies *Thurston's condition* if each  $\alpha_i$  is a nontrivial positive permutation braid, and for any  $1 \leq i \leq r-1$  we have that any  $j$  such that  $\alpha_{i+1} = \sigma_j \gamma_{i+1}$  also satisfies  $\alpha_i = \gamma_i \sigma_j$  where  $\gamma_i, \gamma_{i+1} \in S_n^+$ .

**Proposition 2.2** (Left-canonical form of a braid). *Given any braid  $\beta \in B_n$ , there exists a unique decomposition*

$$(2.2) \quad \beta = \Delta^k \alpha_1 \cdots \alpha_r,$$

where  $k \in \mathbb{Z}$  is maximal,  $\alpha_i \in S_n^+$  and the product  $\alpha_1 \cdots \alpha_r$  satisfies Thurston's condition.

### 3. PROOF OF THEOREM 1.1

Our first step consists of looking at the behavior of  $\Delta$  and permutation braids under the antiautomorphism  $\text{rev}$ .

**Lemma 3.1.** *The following properties hold:*

- (1)  $\overline{\Delta} = \Delta$
- (2) *The set of permutation braids is invariant under  $\text{rev} : w \mapsto \overline{w}$ .*
- (3)  $\overline{v^{-1}} = \overline{v}^{-1}$  for all  $v \in B_n$ .

We now identify the basic problem. Let  $\beta = \Delta^k \alpha_1 \cdots \alpha_r$  be the left-canonical form for a braid  $\beta \in B_n$ . We cannot assume that the decomposition  $v = \alpha_1 \cdots \alpha_r$  remains in left-canonical form when viewed in  $v\overline{v}$ . Indeed, after multiplying on the right by the reversed braid, the product  $\alpha_1 \cdots \alpha_r$  (viewed in  $v\overline{v}$ ) may cease to satisfy Thurston's condition. A simple example is provided by

$$\alpha_1 = \sigma_1 \sigma_3, \quad \alpha_2 = \sigma_3.$$

Both  $\alpha_1$  and  $\alpha_2$  are positive permutation braids and the product  $\gamma = \alpha_1 \alpha_2$  satisfies Thurston's condition. However, when we write  $\beta = \alpha_1 \alpha_2 \overline{\alpha_2} \overline{\alpha_1}$  in left-canonical form, we find

$$\beta = \underbrace{\sigma_3 \sigma_1}_{\alpha_1} \underbrace{\sigma_3 \sigma_1}_{\alpha'_2} \underbrace{\sigma_3}_{\alpha''_2} \underbrace{\sigma_3}_{\alpha''_1}$$

so that the second canonical factor  $\alpha'_2$  does *not* coincide with  $\alpha_2$ .

We start by proving the Theorem 1.1 for positive braids.

**3.1. Proof of Theorem 1.1 for positive braid words.** We start by proving the following lemma:

**Lemma 3.2.** *Let  $v\overline{v} = \sigma w' \overline{w'} \sigma$  and let  $v\overline{v} \equiv w_0 = w_1 = \cdots = w_k \equiv \sigma v'$  be a sequence of positive braid words such that each  $w_{i+1}$  is the outcome of the activation of one relation out of the relations in the semigroup  $B_n^+$  on  $w_i$  according to Jacquemard's algorithm. Then, all relations are performed only within the first half of the word  $w_i$  which implies they all involve only letters from  $v$ .*

*Proof.* Notice that since  $v\overline{v} = \sigma w' \overline{w'} \sigma$ , the success of the Jacquemard's algorithm is guaranteed. Hence we know that  $\sigma$  is one of the letters of  $v$ . Now, we need to prove that in each step of Jacquemard's algorithm that uses a relation, it occurs in the first half of the word  $v$ .

For step (1) of the algorithm, this is obvious: all relations involve  $\sigma$  and left neighbors of  $\sigma$ ; since  $\sigma$  is in  $v$ , all relations occur inside  $v$ . Moreover, relations can be activated mirror like on  $\overline{v}$  as well; Hence, we maintain the

palindromic structure of the word. This implies that when we need to move to step (2) of the algorithm we have

$$v\bar{v} = v_1\tau\sigma v_2\bar{v} = v_1\tau\sigma v_2\bar{v}_2\sigma\tau\bar{v}_1,$$

where  $\tau$  and  $\sigma$  do not commute, and  $\sigma$  is not in  $v_1$ .

If this is the case, the algorithm calls itself recursively using  $v_2\bar{v} = v_2\bar{v}_2\sigma\tau\bar{v}_1$  and  $\tau$ , trying to extract  $\tau$  to the left, first by looking for the leftmost  $\tau$  letter in  $v_2\bar{v} = v_2\bar{v}_2\sigma\tau\bar{v}_1$ .

Assume by contradiction that  $\tau$  is not a letter of  $v_2$  (hence is not a letter of  $\bar{v}_2$ ). Then the leftmost  $\tau$  letter in  $v_2\bar{v} = v_2\bar{v}_2\sigma\tau\bar{v}_1$  appears to the right of  $\sigma$  and to the left of  $\bar{v}_1$ . In order to extract this  $\tau$  to the left of  $v_2$ , we need to activate another recursive call of the algorithm on  $\bar{v}_1$  with the letter  $\sigma$  (since in our case  $\sigma$  and  $\tau$  do not commute and we have to use step (2) of the algorithm). But,  $\sigma$  is not in  $\bar{v}_1$  (since it is not in  $v_1$ ). Therefore  $\sigma$  cannot be extracted to the left of  $\bar{v}_1$ . This implies that Jacquemard's algorithm failed, and this is a contradiction.

If  $\tau$  is found within  $v_2$  and during the process of extracting it to the left we do not encounter the need to use letters from the right half of  $v\bar{v}$  we are finished. Therefore, assume by contradiction that at some point in the process we encounter a relation involving a letter of  $\bar{v}$ . Again, since until this step all relations were activated only in the left half of the word their mirror image can be activated on the right half of the word, so the palindromic structure of the word is preserved.

Suppose that we performed  $k$  recursive steps of the algorithm. Then, our word looks like:

$$v\bar{v} = v_1\tau_1\sigma v_2\tau_2\tau_1v_3\tau_3\tau_2 \cdots v_{k-1}\tau_{k-1}\tau_{k-2}v_k\tau_k\tau_{k-1}v_{k+1} \cdot \\ \overline{v_{k+1}\tau_{k-1}\tau_k\bar{v}_k\tau_{k-2}\tau_{k-1}v_{k-1} \cdots \tau_2\tau_3\bar{v}_3\tau_1\tau_2\bar{v}_2\sigma\tau_1\bar{v}_1}$$

where  $\tau_i$  and  $\tau_{i+1}$  are two non commuting letters,  $\sigma$  does not commute with  $\tau_1$  and is not a letter of  $v_1$ . Moreover,  $\tau_i$  is not a letter of  $v_{i+1}$  for any  $i = 1, \dots, k$ .

Now, in this recursion step, we have called the algorithm with the letter  $\tau_k$  and the word  $v_{k+1}\bar{v}_{k+1}\tau_{k-1}\tau_k\bar{v}_k\tau_{k-2}\tau_{k-1}v_{k-1} \cdots \tau_2\tau_3\bar{v}_3\tau_1\tau_2\bar{v}_2\sigma\tau_1\bar{v}_1$ . However, since  $\tau_k$  is not a letter of  $v_{k+1}$  and of  $\bar{v}_{k+1}$  the leftmost  $\tau_k$  in this recursion call is to the right of  $\tau_{k-1}$  and to the left of  $\bar{v}_k$ . Since  $\tau_{k-1}$  does not commute with  $\tau_k$ , another recursion call is needed with the letter  $\tau_{k-1}$  and the word  $\bar{v}_k\tau_{k-2}\tau_{k-1}v_{k-1} \cdots \tau_2\tau_3\bar{v}_3\tau_1\tau_2\bar{v}_2\sigma\tau_1\bar{v}_1$ . Again,  $\tau_{k-1}$  is not a letter of  $\bar{v}_k$  hence the leftmost  $\tau_{k-1}$  in this recursion call is to the right of  $\tau_{k-2}$  and to the left of  $\bar{v}_{k-1}$ . Similarly  $\tau_{k-2}$  does not commute with  $\tau_{k-1}$ , so we continue  $k-3$  recursion calls until we reach a recursion call with the letter  $\tau_1$  and the word  $\bar{v}_2\sigma\tau_1\bar{v}_1$ . Since  $\tau_1$  is not a letter in  $\bar{v}_2$ , the leftmost  $\tau_1$  in this call is to the right of  $\sigma$  and to the left of  $\bar{v}_1$ . This implies that another recursion call is needed in order to extract the letter  $\sigma$  from the word  $\bar{v}_1$ . However, this contradicts the hypothesis on  $v_1$ .

This concludes the proof of all cases, hence all relations are activated inside the left half of the word  $v\bar{v}$  as claimed.  $\square$

Now we are ready to prove the theorem for positive braid words.

**Theorem 3.3.** *Let  $\beta, \beta' \in B_n^+$  be two palindromic positive braids of even length such that  $\beta = v\bar{v}$  and  $\beta' = w\bar{w}$  for some braids  $v, w \in B_n^+$ . Then,  $\beta = \beta'$  in  $B_n$  if and only if  $v = w$  in  $B_n$ .*

*Proof.* By induction on the length  $l = |w|$  of  $w$ . Assume that  $w \equiv \sigma w'$  where  $w' \in B_n^+$ , i.e.,  $\sigma$  is the first letter in  $w$ . Then,  $w\bar{w} = \sigma w' \overline{\sigma w'}$ . This means that  $v\bar{v}$  can be written such that its first letter is  $\sigma$ , that is,  $v\bar{v} = \sigma v' \overline{\sigma v'}$  for some  $v' \in B_n^+$ .

By the embedding theorem of Garside [4] it follows that there is a sequence of words  $v\bar{v} \equiv w_0 = w_1 = \dots = w_k \equiv \sigma v'$ , such that each  $w_{i+1}$  is obtained from  $w_i$  by activating one relation out of the relations in the semigroup  $B_n^+$ .

One possible sequence is the one which uses the relations suggested by the algorithm of Jacquemard given in [5]. Now, Lemma 3.2 shows that every relation used in the sequence is fully contained in  $v$  (the left half of the word), and does not effect  $\bar{v}$ . Therefore, it is possible to activate all relations described in the sequence in a mirror-like image on  $\bar{v}$  and get  $w\bar{w} = \sigma w' \overline{\sigma w'} = v\bar{v} = \sigma v' \overline{\sigma v'} = \sigma v'' \overline{\sigma v''}$ .

However, if this is the case  $w' \overline{w'} = v'' \overline{v''}$  where  $|w' \overline{w'}| = |v'' \overline{v''}| = l - 2$ . Therefore, by the induction hypothesis we have  $w' = v''$  which implies that  $\sigma w' = w = v = \sigma v''$ . □

Next, we use the above, and give the proof of Theorem 1.1.

**3.2. Proof of Theorem 1.1 for the general case.** Let  $v \in B_n$  and set  $\beta = v\bar{v}$ . We need to prove that if  $\beta = w\bar{w}$ , then  $v = w$ .

Let  $v = \Delta^k \alpha_1 \dots \alpha_r$  be the left-canonical form of  $v$ . Then  $\bar{v} = \bar{\alpha}_r \dots \bar{\alpha}_2 \bar{\alpha}_1 \Delta^k$  (Lemma 3.1(1)). Hence,

$$\beta = v\bar{v} = \Delta^k \alpha_1 \dots \alpha_r \bar{\alpha}_r \dots \bar{\alpha}_1 \Delta^k.$$

Moreover, let  $w = \Delta^j \beta_1 \dots \beta_p$  be the left-canonical form of  $w$ . We have

$$\beta = w\bar{w} = \Delta^j \beta_1 \dots \beta_p \bar{\beta}_p \dots \bar{\beta}_1 \Delta^j$$

(Note that it is not necessary that  $r = p$ ).

Without loss of generality, we may assume that  $j < k < 0$  (Otherwise,  $\beta \in B_n^+$  and we use Theorem 3.3). By multiplying  $\beta$  by  $\Delta^{-j}$  on the left and on the right, we obtain:

$$\Delta^{-j} \beta \Delta^{-j} = \Delta^{-j} v \bar{v} \Delta^{-j} = \Delta^{k-j} \alpha_1 \dots \alpha_r \bar{\alpha}_r \dots \bar{\alpha}_1 \Delta^{k-j},$$

and

$$\Delta^{-j} \beta \Delta^{-j} = \Delta^{-j} x \bar{x} \Delta^{-j} = \beta_1 \dots \beta_p \bar{\beta}_p \dots \bar{\beta}_1.$$

These are two equal positive braid words and have the form  $v' \overline{v'} = w' \overline{w'}$ , where  $v' = \Delta^{k-j} \alpha_1 \dots \alpha_r$  and  $w' = \beta_1 \dots \beta_p$  respectively. Therefore, Theorem 3.3 applies and we conclude that  $v = w$ . □

As a consequence, we obtain the following corollaries:

**Corollary 3.4.** *Let  $\beta = x\bar{x} \in B_n^+$  be a positive palindromic braid of even length, and let the left-canonical normal form of  $x$  be  $\alpha_1 \dots \alpha_r$  such that*

$\alpha_1 \neq \Delta$ . Then, the left-canonical normal form of  $x\bar{x}$  is  $\beta_1 \cdots \beta_p$  where  $\beta_1 \neq \Delta$ .

*Proof.* Otherwise because the process of Jacquemard's algorithm may be used to transform the word  $\alpha_1 \cdots \alpha_r \bar{\alpha}_r \cdots \bar{\alpha}_1$  into its left-canonical form, and since it extracts letters to the left only from the first half of the word, we might have

$$\beta = \alpha_1 \cdots \alpha_r \bar{\alpha}_r \cdots \bar{\alpha}_1 = \Delta \gamma_1 \cdots \gamma_q \bar{\gamma}_q \cdots \bar{\gamma}_1 \Delta,$$

which means, by Theorem 3.3, that  $\alpha_1 \cdots \alpha_r = \Delta \gamma_1 \cdots \gamma_q$ . By the uniqueness of the left-canonical normal form, we deduce that  $\alpha_1 = \Delta$ , which is a contradiction.  $\square$

We may generalize Corollary 3.4:

**Corollary 3.5.** *Let  $v \in B_n^+$  be a positive braid, and let  $n(v)$  denote the number of leading permutation braids which are  $\Delta$  when  $v$  is written in its left-canonical normal form. Then, for  $\beta = v\bar{v}$ , we have  $n(v\bar{v}) = 2n(v)$ .*

*Proof.* Since  $v$  is written in left-canonical form as  $\Delta^{n(v)} \alpha_1 \cdots \alpha_r$  where  $\alpha_1 \neq \Delta$ , we have that  $\beta = \Delta^{n(v)} \alpha_1 \cdots \alpha_r \bar{\alpha}_r \cdots \bar{\alpha}_1 \Delta^{n(v)}$ . Note that since  $\Delta$  almost commutes with any permutation braid, we may write

$$\beta = \Delta^{2n(v)} \alpha'_1 \cdots \alpha'_r \bar{\alpha}'_r \cdots \bar{\alpha}'_1,$$

where  $\alpha'_i = \alpha_i$  if  $n(v)$  is even and  $\alpha'_i$  is obtained from  $\alpha_i$  by replacing each  $\sigma_j$  by  $\sigma_{n-j}$  in case that  $n(v)$  is odd. In any of the cases, the product  $\alpha'_1 \cdots \alpha'_r$  keeps its left-canonical form. Hence, using the same argument as in Corollary 3.4 we get that  $n(v\bar{v}) = 2n(v)$ .  $\square$

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FLORIAN DELOUP, EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM, THE HEBREW UNIVERSITY OF JERUSALEM, 91904 JERUSALEM, ISRAEL, and

LABORATOIRE EMILE PICARD, UMR 5580 CNRS/UNIVERSITÉ PAUL SABATIER, 118  
ROUTE DE NARBONNE, 31062 TOULOUSE, FRANCE.

*E-mail address:* `deloup@picard.ups-tlse.fr`

DAVID GARBER, EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS,  
GIVAT RAM, THE HEBREW UNIVERSITY OF JERUSALEM, 91904 JERUSALEM, ISRAEL, *and*

DEPARTMENT OF SCIENCES, HOLON ACADEMIC INSTITUTE OF TECHNOLOGY, 52 GOLOMB  
STREET, 58102 HOLON, ISRAEL.

*E-mail address:* `garber@math.huji.ac.il`

*E-mail address:* `garber@hait.ac.il`

SHMUEL KAPLAN, DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY RAMAT-  
GAN 52900, ISRAEL.

*E-mail address:* `kaplansh@macs.biu.ac.il`

MINA TEICHER, DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY RAMAT-GAN  
52900, ISRAEL.

*E-mail address:* `teicher@macs.biu.ac.il`